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REPORT

SOLENOID ERRORS DUE TO YAW, ECCENTRICITY,
AND INCLINATION

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Abstract

A common method of measuring the velocity of a magnetized projectile is to fire it through short "solenoid" coils. The induced voltages actuate suitable chronographic equipment. If the magnetized projectile traverses a coil at a distance b from its center, the "cross-over" of its signal will occur approximately $(b/2v)\left[\sin(\theta + \delta) + \sin\theta\right]$ seconds later than it would were there no yaw, eccentricity, or inclination. In this formula v is the velocity, δ is the meridional yaw, and θ is the meridional inclination of the trajectory to the axis of the coil.

1. General. Circular coils of wire called "solenoids," are frequently employed to indicate the passage through them of longitudinally magnetized projectiles. The induced electromotive forces are allowed to actuate oscillographs, or chronographs of other types, in order to furnish the elapsed time between coils. Such instrumentation is often used for measuring velocity and retardation. It is well known that time lags in the recording equipment may cause the measured time interval to differ from the true time interval between coils, and efforts are made in practice to overcome, or diminish, such effects. Studies of such time lags in recording equipment in relation to the electrical circuit conditions have been made by Kent[1] and by Vinti[2]. However, although it has also been realized that yaw, eccentricity, and inclination of a projectile with respect to a coil can introduce errors, there does not seem to have been any general and quantitative evaluation of them. Since it is desirable to be able to predict the accuracy of solenoid coil equipment, simple and approximate formulas for the errors due to yaw, eccentricity, and inclination are given here. Their derivation will be found in an Appendix. Although only approximate, the formulas are sufficiently accurate to constitute a guide as to the degree of care with which guns and solenoid coils should be aligned and oriented, and to facilitate the critical evaluation of experimental data.

2. Discussion. When a longitudinally magnetized projectile approaches a circular coil, traverses it, and recedes from it, the changing magnetic flux induces an electromotive force V in the coil. The geometrical conditions of passage through the coil may be said to be standard, or ideal, when the projectile moves along the axis of the coil and at the same time the axis of the projectile remains perpendicular to the plane of the coil. One is here interested in the changes in V caused

[1] R. H. Kent, Ordnance Technical Note No. 8 (1927)

[2] J. P. Vinti, BRL Report No. 271, "Some Developments in the Theory of the Solenoid Chronograph;" Vinti and A. H. Hodge, Addendum to BRL Report No. 271, "Experimental Determination of Solenoid e.m.f. Curves;" Vinti, BRL Report No. 305, "The Practical Determination of Lag in the Solenoid Chronograph."

by small departures from the preceding standard conditions. Time lags, if there are any, in the recording mechanism will not be considered here, for one is interested only in the change produced in the recorded signal by small departures from the standard conditions of passage. It is clear that whatever the time lag may be under standard conditions, between the V signal and its record, any time lag in the V signal itself will give rise to substantially the same time lag in the record, over and above the time lag present in the record under standard conditions.

When a magnetic dipole whose axis is perpendicular to a circular coil moves with velocity v along the coil's axis, the resulting electromotive force V is known to be proportional to

$$vz/(a^2 + z^2)^{5/2}$$

where a is the radius of the coil and z is the distance of the dipole from the plane of the coil. A longitudinally magnetized projectile is not exactly a dipole, and the voltage induced by a projectile under standard conditions does not follow the preceding law precisely. The dominant term, however, in the external magnetic field of a longitudinally magnetized projectile is the dipole term, and hence the change in the V of a real projectile, produced by a small departure from standard geometrical conditions is approximately equal to the change in the V of an ideal dipole, caused by the same small departure from standard geometrical conditions.

The preceding expression for V shows that V vanishes for a dipole under standard geometrical conditions when z is zero. If, under certain non-standard geometrical conditions, V vanishes for a dipole when z has the value z_1, then it follows that the V of an actual projectile would vanish under such non-standard conditions at a time later, by z_1/v seconds, than it would under standard conditions. Thus, if one measures on an oscillograph record the instant when V, as affected by time lags, vanishes then such an instant will be late by z_1/v seconds as compared with the instant when V would have been observed to vanish in the absence of eccentricity, yaw, and inclination. Alternatively, if the chronographic equipment records the instant when d^2V/dt^2 vanishes, t being the time, then to determine the error in such times

caused by departures from the ideal geometrical conditions, it is sufficient to determine the value of z for which d^2V/dt^2 would vanish for a dipole under the actual geometrical conditions. Under standard geometrical conditions, d^2V/dt^2 for the dipole vanishes when z is zero. A type of equipment that records the instant when d^2V/dt^2, rather than V, vanishes is the electronic counter. There the signal is amplified and differentiated, and the counting mechanism is actuated by the maxima or minima of dV/dt.

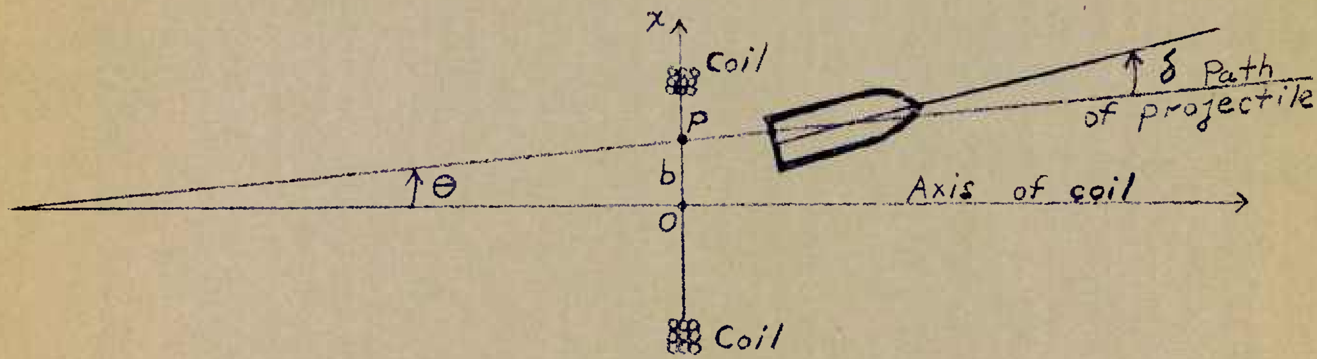
3. Results. Denote by P the point where the magnetic center of the projectile pierces the plane of the coil, and denote by b the distance between P and the center, O, of the coil. The distance b may be called the "eccentricity." Choose as the z-axis the axis of the coil, in the direction of motion; choose as the x-axis the line OP; choose as the y-axis a line perpendicular to the x- and z-axes, so as to form a right-handed system. Denote the direction cosines of the direction of motion of the projectile by l, m, n, and the direction cosines of the axis of the projectile by l', m', n'. The standard geometrical conditions are $b = 0$, $l = m = l' = m' = 0$, and $n = n' = 1$. Then under the actual, non-standard conditions, it is shown in the Appendix that the signal will be late by the time interval

$$\Delta t = (b/2v)\ (l + l') \tag{1}$$

approximately, and this is the desired formula.

The results may be expressed in an alternative form. Project the path of the projectile, and the axis of the projectile, upon the plane xz. Denote by θ the angle between the z-axis and the projection of the path, and by $\delta + \theta$ the angle between the z-axis and the projection of the projectile's axis. One may call θ the meridional component of the inclination, and δ the meridional component of the angle of yaw. Then (1) may be written

$$\Delta t = (b/2v)\left[\sin(\delta + \theta) + \sin\theta\right]. \tag{2}$$



As a numerical illustration, if δ is 6°, and θ is 2° while b is 0.5 feet, then by (2) v Δt is 0.043 feet. If the remaining velocity is 3000 f/s, then Δt is 0.000014 seconds, approximately.

The preceding equations are applicable whether the chronographic equipment is designed to record the instants when the signals are zero, or the instants when their rates of change are largest.

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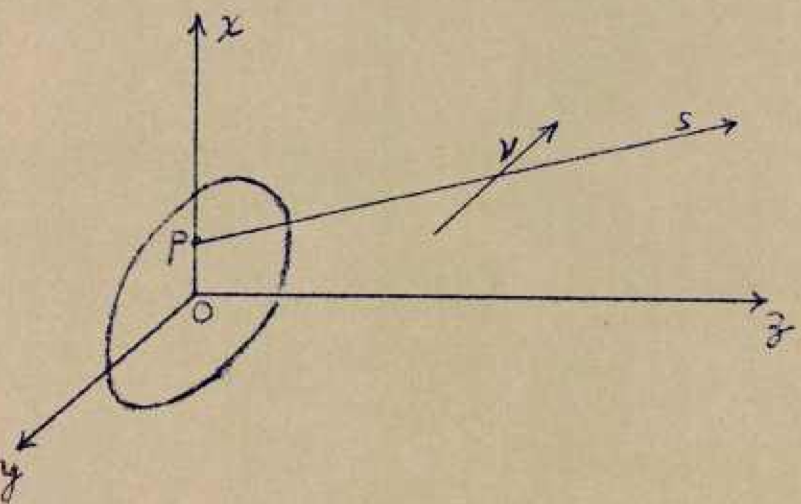
APPENDIX

The EMF Induced in a Circular Coil By a Moving Dipole.

Take a right-handed set of rectangular axes as in paragraph 3 of this Report. The radius of the coil is a. The origin, O, is at the center of the coil; the axis Oz is the axis of the coil, with its positive sense in the direction of motion; the axes Ox and Oy are in the plane of the coil, and are so chosen that the coordinates of the point P, where the path of the dipole intersects the plane of the coil are b, 0, 0. One may call b the "eccentricity." It is sufficiently accurate to consider the path to be a straight line, in the vicinity of the coil, with direction cosines l, m, n. Distance along this line, measured from P, is denoted by s. The strength of the dipole is denoted by μ, the direction cosines of its axis by l', m', n', and distance along the axis of the dipole, measured from the dipole, is denoted by ν.



The solid angle, Ω, subtended by the coil at a point Q is a function Ω (z, r) of the cylindrical coordinates z and r of Q where $r^2 = x^2 + y^2$. The solid angle Ω is so defined as to be zero when Q is the point 0,0, $-\infty$. Then it is 2π at the origin and 4π at the point 0, 0, $+\infty$.

The function Ω (z, r) can be derived thus. At a point on the axis, the solid angle is clearly

$$\Omega = 2\pi\left(1 + \frac{z}{(a^2 + z^2)^{1/2}}\right) \qquad (3)$$

since the radius of a spherical surface, whose center is the point Q and which contains the coil, is $(a^2 + z^2)^{1/2}$. This expression can be expanded in powers of z by the binomial theorem; and for $z < a$,

$$\Omega = 2\pi\left[1 + \frac{z}{a} - \frac{1}{2}\frac{z^3}{a^3} + \frac{3}{8}\frac{z^5}{a^5} - \frac{5}{16}\frac{z^7}{a^7} + \ldots\right], \qquad (4)$$

while for $z > a$ there is another expression. From (4) it is possible to deduce the solid angle at points not on the axis. Take spherical polar coordinates, with the center of the coil as origin, the axis of the circle as the line $\theta = 0$, and with the symbol R denoting the distance from the center of the circle. Since the desired solid angle Ω at any point is equal to the magnetic potential at that point that would be produced by a unit current flowing in the circle, it follows that Ω must be a solution of Laplace's equation $\nabla^2\Omega = 0$. Inside the sphere $R = a$, the solid angle Ω is also symmetrical about the axis $\theta = 0$, and remains finite at the origin. It is therefore capable of expansion in the form

$$\Omega = \sum_{n=0}^{\infty} A_n R^n P_n(\cos\theta) \qquad (5)$$

where the A_n's are numerical coefficients, and $P_n(\cos\theta)$ is the Legendre coefficient of order n and argument $\cos\theta$. Now

$$P_n(1) = 1,$$

so along the axis $\theta = 0$ equation (5) becomes

$$\Omega = \sum A_n R^n \qquad (6)$$

enabling the coefficients A_n to be determined by comparison of equations (4) and (6), since $R = z$ for points on the axis.

It is seen that the even orders above the zero'th are absent, and since

$$\cos \theta = z/R$$

while

$$P_1(x) = x,$$

$$P_3(x) = (1/2)(5x^3 - 3x),$$

$$P_5(x) = (1/8)(63x^5 - 70x^3 + 15x),$$

$$P_7(x) = (1/16)(429x^7 - 693x^5 + 315x^3 - 35x),$$

one finds that when $R < a$,

$$\Omega = 2\pi \left[1 + \frac{R}{a}\frac{z}{R} - \frac{R^3}{4a^3}\left(5\frac{z^3}{R^3} - 3\frac{z}{R}\right) + \frac{3R^5}{64a^5}\left(63\frac{z^5}{R^5} - 70\frac{z^3}{R^3} + 15\frac{z}{R}\right) - \ldots\right].$$

Making use of the relation

$$R^2 = z^2 + r^2$$

one finds that

$$\Omega(z,r) = 2\pi \left[1 + \frac{z}{a} - \frac{1}{4a^3}(2z^3 - 3zr^2) + \frac{3}{64a^5} \cdot (8z^5 - 40z^3 r^2 + 15zr^4) - \ldots\right] \qquad (7)$$

which is the desired function, valid when $R < a$. An expansion $\Omega(z,r)$ that is valid when $R > a$ can be similarly derived, but is not needed here.

At any instant, the surface integral over the area of the coil of the normal component of magnetic induction is

$$N = \mu \frac{\partial \Omega}{\partial \nu}$$

$$= \mu \left(\ell' \frac{\partial \Omega}{\partial x} + m' \frac{\partial \Omega}{\partial y} + n' \frac{\partial \Omega}{\partial z} \right)$$

$$= \mu \left[\left(\ell' \frac{\partial r}{\partial x} + m' \frac{\partial r}{\partial y} \right) \frac{\partial \Omega}{\partial r} + n' \frac{\partial \Omega}{\partial z} \right]$$

and since

$$\frac{\partial r}{\partial x} = \frac{x}{r} ; \quad \frac{\partial r}{\partial y} = \frac{y}{r} ,$$

it follows that

$$N = \mu \left[(\ell' x + m' y) \frac{1}{r} \frac{\partial \Omega}{\partial r} + n' \frac{\partial \Omega}{\partial z} \right] , \qquad (8)$$

$$= N(x, y, r, z),$$

an explicit function of the four non-independent variables x, y, r, z among which r is a function of x and y.

The induced electromotive force is given by

$$V = -T \frac{dN}{dt}$$

$$= - vT \frac{dN}{ds}$$

or

$$V = - vT \left(\ell \frac{\partial N}{\partial x} + m \frac{\partial N}{\partial y} + n \frac{\partial N}{\partial z} \right) \qquad (9)$$

where T is the number of turns in the coil, v is the velocity of the dipole, and N is regarded as a function of the three independent variables, x, y, z. If N is available as an explicit function N(x,y,r,z) of the for non-independent variables x, y, r, z then equation (9) becomes

$$V = -vT\left[\ell\frac{\partial N}{\partial x} + m\frac{\partial N}{\partial y} + (\ell x + my)\frac{1}{r}\frac{\partial N}{\partial r} + n\frac{\partial N}{\partial z}\right] \qquad (10)$$

and hence, by equation (8),

$$V = -\mu vT\Big\{(\ell\ell' + mm')\frac{1}{r}\frac{\partial\Omega}{\partial r} + (\ell x + my)(\ell' x + m'y)\frac{1}{r}\frac{\partial}{\partial r}\frac{1}{r}\frac{\partial\Omega}{\partial r}$$

$$+\left[n'(\ell x + my) + n(\ell' x + m'y)\right]\frac{1}{r}\frac{\partial^2\Omega}{\partial z\partial r} + nn'\frac{\partial^2\Omega}{\partial z^2} \quad . \qquad (11)$$

From equation (7), the leading terms in the partial derivatives above may be evaluated. They are

$$\frac{1}{r}\frac{\partial\Omega}{\partial r} = +2\pi\left(\frac{3z}{2a^3} - \frac{15}{4}\frac{z^3}{a^5} + \frac{45zr^2}{16a^5} + \ldots\right),$$

$$\frac{1}{r}\frac{\partial}{\partial r}\frac{1}{r}\frac{\partial\Omega}{\partial r} = +2\pi\left(\frac{45}{8}\frac{z}{a^5} + \ldots\ldots\ldots\right),$$

$$\frac{1}{r}\frac{\partial^2\Omega}{\partial z\partial r} = +2\pi\left(\frac{3}{2a^3} - \frac{45}{4}\frac{z^2}{a^5} + \frac{45}{16}\frac{r^2}{a^5} + \ldots\right),$$

$$\frac{\partial^2\Omega}{\partial z^2} = -2\pi\left(\frac{3z}{a^3} - \frac{15}{2}\frac{z^3}{a^5} + \frac{45}{4}\frac{zr^2}{a^5} + \ldots\right).$$

Near the instant of passage through the coil $\underline{l}$, $\underline{l}'$, $\underline{m}$, and $\underline{m}'$ are small quantities, of the order of magnitude of the yaw; usually no larger than about 1/6 (for 10° yaws). The quantity $\underline{x}$ is nearly equal to $\underline{b}$, and $\underline{y}$ is closely zero; while $\underline{n}$ and $\underline{n}'$ are approximately unity. It follows from the preceding relations that at the instant of passage, when $\underline{z}$ is zero, the only contribution to $\underline{V}$ comes from the terms involving

$$\frac{1}{r}\frac{\partial^2\Omega}{\partial z\partial r}$$

and thus at this instant

$$V = -3\pi\mu vT'b(\underline{l} + \underline{l}')/a^3$$

approximately while the principal term in dV/dz is

$$6\pi\mu vT/a^3.$$

Hence $\underline{V}$ is zero at the instant when

$$\underline{z} = \frac{b}{2}(l + l') \qquad (12)$$

approximately, which occurs approximately

$$\Delta t = (b/2v)(\underline{l} + \underline{l}') \qquad (13)$$

seconds after passage. This is the desired relation.

A word is desirable as to the degree of approximation involved in equations (12) and (13). In deriving expressions for $\underline{V}$ and dV/dz from equations (7) and (11) only leading terms have been retained, and terms containing the factors $(b/a)^2$, $(b/a)^4$, etc., have been discarded. A rough evaluation of the effect of neglected terms indicates that the right-hand members of equations (12) and (13) are correct when b/a is very small, and should be multiplied by 0.7 when b/a is 0.5, and by 0.4 when b/a is unity. Thus the formulas in this Report should be regarded as furnishing the order of magnitude, only, of errors due to yaw, eccentricity, and inclination. Although it would be possible to retain terms that have been discarded, the resulting gain in accuracy would be largely formal, rather than real, because of the magnetic differences between actual projectiles and ideal dipoles; and the resulting formulas would be considerably more complicated than the present formulas. The present formulas (1), (2), (12), and (13) are sufficiently accurate as they stand for the purposes for which they are intended, which are to furnish a guide as to the degree of care with which guns and velocity-measuring equipment should be aligned and to facilitate the critical evaluation of experimental data.

Equation (13) gives the time interval, Δt, between the passage of the dipole through the plane of the coil and the instant when the induced electromotive force, V, is zero. In case the chronographic equipment differentiates

the voltage input , and determines the instant when the derivative is a maximum or minimum, the time interval that is of interest is the interval $\Delta \underline{t}'$ between coil passage and the vanishing of d^2V/dt^2. This quantity vanishes, substantially, when d^2V/dz^2 vanishes. At the instant of coil passage,

$$d^2V/dz^2 = 45\pi\mu vI\ b(\underline{1} + \underline{1}')/a^5$$

from equation (11); and the principal term in d^3V/dz^3 is

$$-90\pi\mu vI/a^5.$$

Accordingly, d^2V/dz^2 is zero when

$$z = (b/2)(\underline{1} + \underline{1}')$$

approximately, and thus

$$\Delta t' = (b/2v)(\underline{1} + \underline{1}')$$

seconds, which shows that $\Delta t' = \Delta t$ approximately.